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Canonical solution of the state labelling problem for $SU(n) \supset SO(n)$ and Littlewood's branching rule: III. $SU(3) \supset SO(3)$ case

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Abstract. The first two papers in the present series discussed in general terms a new solution to the state labelling problem for the d-row irreducible representations (irreps) of $SU(n)$, when reduced with respect to $SO(n)$. This solution was termed canonical because it reflects the operation of Littlewood's branching rule for $U(n) \supset O(n)$ in a straightforward way. In the present paper, the $SU(3) \supset SO(3)$ case is worked out in detail. Explicit expressions of the canonical basis states of SO(3) irreps L belonging to an SU(3) irrep $[h_1 h_2]$ are obtained. The matrix of the transformation from the Bargmann–Moshinsky basis to the canonical one is also calculated. It is shown that in both bases the extra label necessary to completely specify the states can be chosen as the label j_s characterising an 'intermediate' SU(2) irrep in the reduction of the product representation $j_L \times j_s$ into SU(2) irreps j, where $j = \frac{1}{2}(h_1 - h_2)$ and $j_L = \frac{1}{2}L$ or $\frac{1}{2}(L-1)$ whenever $h_1 + h_2 - L$ is even or odd.

1. Introduction

The purpose of this series of papers is to present a new solution to the state labelling problem for the d-row irreducible representations (irreps) of $SU(n)$, when reduced with respect to $SO(n)$. The first two papers (henceforth referred to as I and II and whose equations will be subsequently quoted by their number preceded by I or II) respectively dealt with the cases where n is arbitrary and $d \leq [\frac{1}{2}n]$ (Deenen and Quesne 1983) or $d > [\frac{1}{2}n]$ (Quesne 1984). In these papers, the proposed solution was termed canonical because it reflects in a straightforward way the reduction of the internal state labelling problem for $U(n) \supset O(n)$ to the external state labelling problem for $U(d)$, as expressed in Littlewood's branching rule (1950).

Papers I and II were concerned with some general structural results for the basis states of $O(n)$ irreps belonging to a given $U(n)$ irrep, but did not give any of their explicit expressions. With the present paper, we would like to fill in this gap. Since the basis states involve some $U(d)$ Wigner (and sometimes also recoupling) coefficients, for practical purposes we have to restrict ourselves to cases where the $U(d)$ Wigner–Racah algebra is known. Therefore we consider the case of SU(3), for which the general irreps are two-row ones and only require the knowledge of ordinary SU(2) Wigner coefficients.

Our purpose is twofold: firstly to show that the canonical solution is as tractable as other solutions to the state labelling problem while having a deeper group-theoretical

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significance; secondly to relate the canonical basis to other ones, especially to the Bargmann–Moshinsky basis (1961), also based upon Littlewood's branching rule through the use of the elementary permissible diagram method (Moshinsky and Syamala Devi 1969, Sharp and Lam 1969).

In § 2, we derive an explicit expression for the canonical basis states of the irreps L of SO(3) contained in a given irrep $[h_1h_2]$ of SU(3). In § 3, we rewrite the Bargmann–Moshinsky basis in Bargmann representation (1961) in terms of the same variables used for the canonical basis. Finally in § 4, we obtain the matrix of the transformation relating the Bargmann–Moshinsky basis to the canonical one.

2. Canonical basis for SU(3) ⊃ SO(3)

In Bargmann representation, the SU(3) generators can be written as

$$C_{st}=\sum_{i=1}^{2} z_{is}\frac{\partial}{\partial z_{it}}, \qquad s,t=1,2,3, \tag{2.1}$$

in terms of six complex variables z_{is}, $i=1,2$, $s=1,2,3$, and the corresponding differential operators $\partial/\partial z_{is}$. The SO(3) subgroup generators in spherical components are defined by

$$L_1=\sum_i\left(b_i\frac{\partial}{\partial c_i}-c_i\frac{\partial}{\partial a_i}\right), \qquad L_0=\sum_i\left(b_i\frac{\partial}{\partial b_i}-a_i\frac{\partial}{\partial a_i}\right), \qquad L_{-1}=\sum_i\left(a_i\frac{\partial}{\partial c_i}-c_i\frac{\partial}{\partial b_i}\right), \tag{2.2}$$

where a_i, b_i, c_i are given by

$$a_i=2^{-1/2}(z_{i1}-\mathrm{i}z_{i2}), \qquad b_i=2^{-1/2}(z_{i1}+\mathrm{i}z_{i2}), \qquad c_i=z_{i3}, \tag{2.3}$$

in accordance with equation (I3.1), where the α index that can only take the value 1 is suppressed.

Definition (2.2) is the standard one for SO(3) and consistent with that used by Bargmann and Moshinsky (1961). It differs however from Wong's convention for SO(n) (1967), as adopted in I and II, by a sign change in the weight generator (H_1 in equation (II3.3) becoming $-L_0$), and the resulting interchange of the raising and lowering generators (E_3^1 in equation (II3.3) becoming $-L_{-1}$). These modifications can easily be taken into account by permuting a_i with b_i in all the results of I and II. For instance, the change of variables (II3.5) becomes

$$u_i=b_i, \qquad i=1,2, \qquad v=c_1, \qquad w_{ij}=w_{ji}=a_ib_j+a_jb_i+c_ic_j, \qquad 1\leqslant i\leqslant j\leqslant 2 \tag{2.4}$$

According to Littlewood's modified branching rule given in equation (II2.3), the multiplicity of the SO(3) irrep L in the SU(3) irrep $[h_1h_2]$ is equal to

$$\sum_{h_1^s h_2^s} g_{[L][h_1^s h_2^s][h_1h_2]} \qquad \text{or} \qquad \sum_{h_1^s h_2^s} g_{[L1][h_1^s h_2^s][h_1h_2]} \tag{2.5}$$

whenever h_1+h_2-L is even or odd respectively. Here $g_{\rho\sigma\tau}$ denotes the multiplicity of the irrep τ of U(3) in the product representation $\rho\times\sigma$, and the summation over h_1^s, h_2^s is restricted to even integers. In § 4 of II it was shown that the highest weight states (HWS) of the equivalent SO(3) irreps L contained in $[h_1h_2]$ can be distinguished by the pattern

$$(\Gamma^s) = \begin{pmatrix} \gamma^s \\ [h_1^s h_2^s] \end{pmatrix}, \tag{2.6}$$

where h_1^s, h_2^s retain the same meaning as in equation (2.5),

$$\gamma^s = h_1 - L, \tag{2.7}$$

and

$$\begin{aligned} h_1^s + h_2^s - \gamma^s &= h_2 && \text{if } h_1 + h_2 - L \text{ is even,} \\ &= h_2 - 1 && \text{if } h_1 + h_2 - L \text{ is odd.} \end{aligned} \tag{2.8}$$

Due to equations (2.7) and (2.8), (Γ^s) only contains one free label which we choose as the integer

$$j_s = \tfrac{1}{2}(h_1^s - h_2^s), \tag{2.9}$$

specifying the SU(2) irrep corresponding to the U(2) irrep $[h_1^s h_2^s]$. In analogy with equation (2.9), we introduce the following notations

$$j = \tfrac{1}{2}(h_1 - h_2), \tag{2.10}$$

and

$$\begin{aligned} j_L &= \tfrac{1}{2}L && \text{if } h_1 + h_2 - L \text{ is even,} \\ &= \tfrac{1}{2}(L-1) && \text{if } h_1 + h_2 - L \text{ is odd.} \end{aligned} \tag{2.11}$$

Since the extra label j_s characterises an 'intermediate' SU(2) irrep in the reduction of the product representation $j_L \times j_s$ into SU(2) irreps j, its allowed values are limited by the usual triangular inequalities (Edmonds 1957). Additional restrictions come from equations (2.7) and (2.8), and the parity of h_1^s, h_2^s. By putting them together, we obtain that the allowed j_s values satisfy the following two conditions

$$j_s \text{ has the parity of } [\tfrac{1}{2}(h_1 + h_2 - L)], \tag{2.12a}$$

$$|j - j_L| \leq j_s \leq \min\{j + j_L, [\tfrac{1}{2}(h_1 + h_2 - L)]\}, \tag{2.12b}$$

where $[\frac{1}{2}(h_1 + h_2 - L)]$ denotes the largest integer contained in $\frac{1}{2}(h_1 + h_2 - L)$.

When $h_1 + h_2 - L$ is even, the HWS is an analytic function in u_i and w_{ij}, which according to equation (II4.1) is given by

$$\begin{aligned} \langle u_i, w_{ij} | [h_1 h_2] j_s L \rangle &= \langle u_i, w_{ij} | [h_1 h_2] \max; (L) \max; (\Gamma^s) \rangle \\ &= \sum_{m_s} \langle j_L j - m_s, j_s m_s | jj \rangle \langle u_i | [2j_L] j_L + j - m_s; (L) \max \rangle \\ &\quad \times \langle w_{ij} | [\tfrac{1}{2}h_s + j_s, \tfrac{1}{2}h_s - j_s] \tfrac{1}{2}h_s + m_s; (0) \max \rangle, \end{aligned} \tag{2.13}$$

where the first factor on the right-hand side is an SU(2) Wigner coefficient, and

$$h^s = h_1^s + h_2^s = h_1 + h_2 - L. \tag{2.14}$$

When $h_1 + h_2 - L$ is odd, equations (II4.5), (II4.6), and (II4.7) show that the HWS can be written as

$$\langle u_i, w_{ij} | [h_1 h_2] j_s L \rangle = (b_1 c_2 - b_2 c_1) \langle u_i, w_{ij} | [h_1 - 1, h_2 - 1] j_s L - 1 \rangle, \tag{2.15}$$

where on the right-hand side the first factor satisfies the following relation

$$(b_1c_2-b_2c_1)^2=u_2^2w_{11}-2u_1u_2w_{12}+u_1^2w_{22}, \tag{2.16}$$

and the second factor is given by an expression similar to equation (2.13) with h_1, h_2, and L respectively replaced by h_1-1, h_2-1, and $L-1$.

It remains for explicit expressions to be found for the polynomials in u_i or w_{ij} appearing on the right-hand side of equation (2.13) and respectively associated with the irreps $[L](L)$ and $[\frac{1}{2}h_s+j_s, \frac{1}{2}h_s-j_s](0)$. The polynomial in u_i is just the Bargmann representation of a Gel'fand basis state (Gel'fand and Tseitlin 1950) for the U(2) irrep $[2j_L]$ built from two boson creation operators, and is given by

$$\langle u_i\|[2j_L]j_L+j-m_s;(L)\max\rangle=[(j_L+j-m_s)!\,(j_L-j+m_s)!]^{-1/2}(u_1)^{j_L+j-m_s}(u_2)^{j_L-j+m_s}. \tag{2.17}$$

According to equation (I4.4), the polynomial in w_{ij} corresponding to $m_s=j_s$ is given, apart from some normalisation coefficient, by

$$\langle w_{ij}\|[\tfrac{1}{2}h_s+j_s,\tfrac{1}{2}h_s-j_s]\tfrac{1}{2}h_s+j_s;(0)\max\rangle\propto(w_{11})^{j_s}(w_{12,12})^{(h_s/2-j_s)/2}, \tag{2.18}$$

where

$$w_{12,12}=w_{11}w_{22}-w_{12}^2. \tag{2.19}$$

For the remaining m_s values, it can be determined from the relation

$$\begin{aligned}\langle w_{ij}\|&[\tfrac{1}{2}h_s+j_s,\tfrac{1}{2}h_s-j_s]\tfrac{1}{2}h_s+m_s;(0)\max\rangle\\&=[(j_s+m_s)!]^{1/2}[(j_s-m_s)!\,(2j_s)!]^{-1/2}\\&\quad\times(C_{21})^{j_s-m_s}\langle w_{ij}\|[\tfrac{1}{2}h_s+j_s,\tfrac{1}{2}h_s-j_s]\tfrac{1}{2}h_s+j_s;(0)\max\rangle,\end{aligned} \tag{2.20}$$

where, according to equation (I4.1), C_{21} reduces to

$$C_{21}=2w_{12}\,\partial/\partial w_{11}+w_{22}\,\partial/\partial w_{12}, \tag{2.21}$$

when acting upon a function of the w_{ij} variables. It is straightforward to show by induction over m_s that

$$\begin{aligned}\langle w_{ij}\|&[\tfrac{1}{2}h_s+j_s,\tfrac{1}{2}h_s-j_s]\tfrac{1}{2}h_s+m_s;(0)\max\rangle\\&\propto[(j_s-m_s)!\,(j_s+m_s)!]^{1/2}(w_{12,12})^{(h_s/2-j_s)/2}\sum_\mu[(m_s+\mu)!\,(j_s-m_s-2\mu)!\,\mu!]^{-1}\\&\quad\times(w_{11})^{m_s+\mu}(2w_{12})^{j_s-m_s-2\mu}(w_{22})^\mu,\end{aligned} \tag{2.22}$$

apart from some m_s independent normalisation coefficient. In equation (2.22), the summation over μ goes over all integers for which the arguments of the factorials are non-negative.

Let us introduce equations (2.17) and (2.22) into equation (2.13). By taking into account that (Edmonds 1957)

$$\langle j_Lj-m_s,j_sm_s|jj\rangle\propto(-1)^{j_L-j+m_s}[(j+j_L-m_s)!\,(j_s+m_s)!]^{1/2}[(j_L-j+m_s)!\,(j_s-m_s)!]^{-1/2}, \tag{2.23}$$

where we have neglected all m_s independent factors, we obtain the following result

$$\langle u_i, w_{ij} | [h_1 h_2] j_s L \rangle$$
$$= (w_{12,12})^{(h_s/2 - j_s)/2} \sum_{m_s \mu} (-1)^{j_L - j + m_s} (j_s + m_s)!$$
$$\times [(j_L - j + m_s)! (m_s + \mu)! (j_s - m_s - 2\mu)! \mu!]^{-1}$$
$$\times (u_1)^{j_L + j - m_s} (u_2)^{j_L - j + m_s} (w_{11})^{m_s + \mu} (2 w_{12})^{j_s - m_s - 2\mu} (w_{22})^{\mu}. \quad (2.24)$$

In equation (2.24), we have chosen the normalisation in a convenient way for subsequent purposes. As a consequence of this choice, the HWS are not normalised to unity. This is unimportant since in any case they are not orthogonal with respect to j_s.

Having derived an explicit expression for the canonical HWS, let us turn to the Bargmann–Moshinsky HWS and rewrite them in Bargmann representation in the next section.

3. Bargmann–Moshinsky basis for SU(3) ⊃ SO(3)

In the Bargmann–Moshinsky basis (1961), the HWS of equivalent SO(3) irreps L contained in a given SU(3) irrep $[h_1 h_2]$ are written as products of powers of polynomials corresponding to the HWS of some elementary permissible diagrams (EPD) (Moshinsky and Syamala Devi 1969). These EPD are listed in table 1, together with the associated polynomials written in terms of boson creation operators in spherical components η_{im}, $i = 1, 2$, $m = +1, 0, -1$, and their determinants $\eta_{ij,mm'} = \eta_{im}\eta_{jm'} - \eta_{im'}\eta_{jm}$. The additional label, used to completely specify the HWS of equivalent irreps L, is the power q of the polynomial t, associated with the EPD characterised by the irreps $[2^2](0)$. The HWS explicit form is given in terms of the EPD polynomials by

$$(\eta_{11})^{L - h_2 + 2q} (\eta_{12,10})^{h_2 - 2q} s^{(h_1 - L - 2q)/2} t^q |0\rangle \qquad \text{if } h_1 - L \text{ is even}, \quad (3.1)$$

and

$$w_+ (\eta_{11})^{L - h_2 + 2q} (\eta_{12,10})^{h_2 - 2q - 1} s^{(h_1 - L - 2q - 1)/2} t^q |0\rangle \qquad \text{if } h_1 - L \text{ is odd},$$

where $|0\rangle$ is the boson vacuum state.

In the Bargmann representation, the cartesian components $\eta_{is} (i = 1, 2,\ s = 1, 2, 3)$ of boson creation operators are represented by z_{is}, and their spherical components

Table 1. Polynomials associated with the EPD HWS and the Bargmann representation of the latter.

$[h_1 h_2] L$	Polynomials	Bargmann representation
$[1]\,1$	η_{11}	$-u_1$
$[2]\,0$	$s = \sum_m (-1)^m \eta_{1m}\eta_{1,-m}$	w_{11}
$[21]\,1$	$w_+ = \sum_m (-1)^m \eta_{12,1m}\eta_{1,-m}$	$u_2 w_{11} - u_1 w_{12}$
$[2^2]\,0$	$t = \sum_{mm'} (-1)^{m+m'} \eta_{12,mm'}\eta_{12,-m-m'}$	$2 w_{12,12}$
$[2^2]\,2$	$(\eta_{12,10})^2$	$u_1^2 w_{22} - 2 u_1 u_2 w_{12} + u_2^2 w_{11}$
$[1^2]\,1$	$\eta_{12,10}$	$-(b_1 c_2 - b_2 c_1)$

$\eta_{i+1}, \eta_{i0}, \eta_{i-1} (i=1,2)$ by $-b_i, c_i, a_i$ $(i=1,2)$ respectively. It is therefore straightforward to write the Bargmann representation of the HWS (3.1) in terms of the variables a_i, b_i, and c_i. To express them in terms of the variables u_i and w_{ij}, we might invert the transformation (2.4). It is however easier to directly apply results of § 2 to the HWS of the EPD. Since for the latter the irrep L is contained only once in the irrep $[h_1 h_2]$, their HWS must coincide in the canonical and in the Bargmann–Moshinsky basis, except for some normalisation factor which is easily found by direct substitution of definition (2.4) for u_i and w_{ij}. In this way we obtain the expressions listed in the third column of table 1. For the HWS of the EPD associated with the irreps $[1^2]$ (1), which is non-analytic in u_i and w_{ij}, we retain the old expression in terms of the a_i, b_i, c_i variables.

Since in the next section we shall determine the transformation from the Bargmann–Moshinsky basis to the canonical one, it is worth asking whether a simple relation exists between the extra labels q and j_s used in either basis. From the definition of q (Moshinsky and Syamala Devi 1969), it is clear that it is linked to the 'intermediate' irrep $[h_1^s h_2^s]$ used in Littlewood's modified branching rule by the relation

$$h_2^s = 2q. \tag{3.2}$$

Equations (2.7), (2.8), and (2.9) enable us to express h_2^s in terms of h_1, h_2, L, and j_s. The desired relation between q and j_s is therefore given by

$$j_s = [\tfrac{1}{2}(h_1 + h_2 - L)] - 2q. \tag{3.3}$$

For subsequent purposes, it is advantageous to characterise the Bargmann–Moshinsky HWS by j_s instead of q. We shall denote their Bargmann representation by $(u_i, w_{ij}|[h_1 h_2] j_s L)$, where we use a round bracket to distinguish it from that of the canonical HWS. When $h_1 + h_2 - L$ is even, it can be written as

$$\begin{aligned}
(u_i, w_{ij}|[h_1 h_2] j_s L) &= (-u_1)^{j+j_L-j_s} (w_{11})^{(j+j_s-j_L)/2} (2w_{12,12})^{(h_s/2-j_s)/2} \\
&\quad \times (u_1^2 w_{22} - 2u_1 u_2 w_{12} + u_2^2 w_{11})^{(j_L+j_s-j)/2} \qquad \text{if } h_1 - L \text{ is even}, \qquad (3.4a) \\
&= (u_2 w_{11} - u_1 w_{12})(-u_1)^{j+j_L-j_s} (w_{11})^{(j+j_s-j_L-1)/2} (2w_{12,12})^{(h_s/2-j_s)/2} \\
&\quad \times (u_1^2 w_{22} - 2u_1 u_2 w_{12} + u_2^2 w_{11})^{(j_L+j_s-j-1)/2} \qquad \text{if } h_1 - L \text{ is odd}, \qquad (3.4b)
\end{aligned}$$

and when $h_1 + h_2 - L$ is odd as

$$(u_i, w_{ij}|[h_1 h_2] j_s L) = -(b_1 c_2 - b_2 c_1)(u_i, w_{ij}|[h_1 - 1, h_2 - 1] j_s L - 1), \tag{3.5}$$

where the second factor on the right-hand side is given by an expression similar to equation (3.4) with h_1, h_2, and L respectively replaced by $h_1 - 1$, $h_2 - 1$, and $L - 1$.

4. Transformation from the Bargmann–Moshinsky basis to the canonical one

In this section, we wish to determine the expansion

$$\langle u_i, w_{ij}|[h_1 h_2] j_s L\rangle = \sum_{j'_s} ([h_1 h_2] j'_s L|[h_1 h_2] j_s L\rangle (u_i, w_{ij}|[h_1 h_2] j'_s L) \tag{4.1}$$

of the canonical HWS, defined in equations (2.24) and (2.15), in terms of the Bargmann–Moshinsky HWS, given in equations (3.4) and (3.5). For such purpose, we have to distinguish between eight cases, according to whether $h_1 + h_2 - L$ and $h_1 - L$ are even or odd, and $j \geqslant j_L$ or $j < j_L$. Since they only differ by small details, we shall derive the expansion in one case and then state the general result.

Let us assume that both h_1+h_2-L and h_1-L are even and $j \geqslant j_L$. Then equation (2.12) shows that the allowed values of j_s and j'_s are

$$j_s = j - j_L + 2a, \qquad j'_s = j - j_L + 2b, \tag{4.2}$$

where

$$a, b = 0, 1, \ldots, \min([j_L], \tfrac{1}{2}h_2). \tag{4.3}$$

Equation (4.1) can be rewritten as

$$\langle u_i, w_{ij} | [h_1 h_2] j - j_L + 2aL \rangle = \sum_b A_b^{(a)} \langle u_i, w_{ij} | [h_1 h_2] j - j_L + 2bL \rangle, \tag{4.4}$$

where $A_b^{(a)}$ denotes the coefficients to be determined.

By introducing equations (2.24) and (3.4a) into equation (4.4) and setting $m_s = j - j_L + \rho$, we obtain the following equation

$$\begin{aligned}
&(w_{12,12})^{(h_s/2-j+j_L)/2-a} \sum_{\mu\rho} (-1)^\rho (2j-2j_L+2a+\rho)! \\
&\qquad \times [\mu!\,\rho!\,(j-j_L+\mu+\rho)!\,(2a-2\mu-\rho)!]^{-1} \\
&\qquad \times (u_1)^{2j_L-\rho}(u_2)^\rho (w_{11})^{j-j_L+\mu+\rho}(2w_{12})^{2a-2\mu-\rho}(w_{22})^\mu \\
&\quad = (-1)^L \sum_b A_b^{(a)} (u_1)^{2j_L-2b}(w_{11})^{j-j_L+b}(2w_{12,12})^{(h_s/2-j+j_L)/2-b} \\
&\qquad \times (u_1^2 w_{22} - 2u_1u_2w_{12} + u_2^2 w_{11})^b,
\end{aligned} \tag{4.5}$$

where the summations over μ and ρ go over all integers for which the factorial arguments are non-negative. We note that the left-hand side of this equation is a $(2a)$-degree polynomial in u_2, while on the right-hand side the term corresponding to a given b value is a $(2b)$-degree polynomial in u_2. Hence

$$A_b^{(a)} = 0 \qquad \text{if } b > a, \tag{4.6}$$

showing that in equation (4.1) j'_s is restricted to those values such that $j'_s \leqslant j_s$. The matrix of the transformation from the Bargmann–Moshinsky basis to the canonical one is therefore triangular.

By expanding the right-hand side of equation (4.5) into powers of u_1, u_2, w_{11}, w_{12}, w_{22}, and equating the coefficients of equal powers on both sides, we obtain the following set of equations for the $a+1$ unknowns $A_b^{(a)}$, $b = 0, 1, \ldots, a$,

$$\begin{aligned}
&(-1)^{L+\rho} 2^{(h_s/2-j+j_L)/2} \sum_{b\sigma} A_b^{(a)} (-2)^b (-4)^{\mu-a-\sigma} b!\,(a-b)! \\
&\qquad \times [\sigma!\,(2b-\rho-2\sigma)!\,(\rho+\sigma-b)!\,(\mu-\sigma)!\,(a-b-\mu+\sigma)!]^{-1} \\
&\quad = (-1)^\rho (2j-2j_L+2a+\rho)!\,[\mu!\,\rho!\,(j-j_L+\mu+\rho)!\,(2a-2\mu-\rho)!]^{-1}.
\end{aligned} \tag{4.7}$$

The number of such equations, equal to the number of μ and ρ values in the range $0 \leqslant \rho \leqslant 2a$, $0 \leqslant \mu \leqslant [a - \frac{1}{2}\rho]$, exceeds the number $a+1$ of unknowns and may therefore be reduced with the purpose of easing the determination of $A_b^{(a)}$.

Let us sum over ρ on both sides of equation (4.7) and use the identities (Edmonds 1957)

$$\sum_n (-1)^n (b-\sigma)![n!(b-\sigma-n)!]^{-1} = \delta_{\sigma,b}, \tag{4.8}$$

and

$$\sum_s (-1)^s (t-s)![s!(x-s)!(z-s)!]^{-1}$$
$$= (t-x)!(t-z)![x!z!(t-x-z)!]^{-1} \qquad \text{if } t \geqslant x \geqslant 0 \text{ and } t \geqslant z \geqslant 0, \tag{4.9}$$

where $n = \rho+\sigma-b$, $s = 2a-2\mu-\rho$, $t = 2z = 2j-2j_L+4a-2\mu$, and $x = 2a-2\mu$. System (4.7) is transformed into the following set of relations

$$\sum_{b=0}^{a} A_b^{(a)}(a-b)![2^b(\mu-b)!]^{-1}$$
$$= (-1)^{L+a+\mu} 2^{(-\frac{1}{2}h_s+j-j_L)/2+2a-2\mu}$$
$$\times (2j-2j_L+2a)!(a-\mu)![\mu!(2a-2\mu)!(j-j_L+\mu)!]^{-1}. \tag{4.10}$$

The number of such equations equals that of μ values in the range $0 \leqslant \mu \leqslant a$, i.e., $a+1$. We have therefore obtained a system of $a+1$ linear equations in $a+1$ unknowns. It is straightforward to show by direct substitution that its solution is given by

$$A_b^{(a)} = (-1)^{L+a-b} 2^{(-h_s/2+j-j_L)/2+a} (2j-2j_L+2a)!!(2j-2j_L+2a+2b-1)!!$$
$$\times [(2a-1)!!\, b!(a-b)!(j-j_L+b)!]^{-1}. \tag{4.11}$$

For such a purpose, we have to use the identity†

$$\sum_p (-1)^p (2l+2p-1)!![(2p)!!(q-p)!(k-q+p)!]^{-1}$$
$$= (-1)^q (2l-1)!!(2l-2k+2q-1)!!$$
$$\times [k!(2q)!!(2l-2k-1)!!]^{-1}, \qquad k, l, q \text{ integer}, \qquad k \leqslant l, \tag{4.12}$$

with the following identifications $k = j-j_L+\mu$, $l = j-j_L+a$, $p = b$, and $q = \mu$.

To obtain the matrix of the transformation from the Bargmann–Moshinsky basis to the canonical one, it only remains to express a and b in equation (4.11) in terms of j_s and j_s' by using equation (4.2). The general result, valid for any values of h_1, h_2, and L, is given by

$$([h_1h_2]j_s'L|[h_1h_2]j_sL\rangle$$
$$= (-1)^{L+(j_s-j_s')/2} 2^{(-h_s/2+3j_s)/2} \{[(j+j_s-j_L)/2]\}!\{[(j_L+j_s-j)/2]\}!$$
$$\times (j_s+j_s'-1)!! [\![(j_L+j_s-j)!\{[(j+j_s'-j_L)/2]\}!\{[(j_L+j_s'-j)/2]\}!$$
$$\times \{(j_s-j_s')/2\}!]\!]^{-1} \qquad \text{if } j_s' = (j_s)_{\min}, (j_s)_{\min}+2, \ldots, j_s,$$
$$= 0 \qquad \text{otherwise}. \tag{4.13}$$

† Equation (4.12) is proved by expanding the identity

$$(1+x)^k(1+x)^{-l-1/2} = (1+x)^{-(l-k)-1/2}, \qquad k, l \text{ integer}, \qquad k \leqslant l,$$

into powers of x and by equating the coefficients of equal powers on both sides.

Here $[x]$ denotes the largest integer contained in x, j and j_L are defined in equations (2.10) and (2.11) respectively, h_s and $(j_s)_{\min}$ are given by

$$h_s = 2[\tfrac{1}{2}(h_1 + h_2 - L)], \tag{4.14}$$

and

$$\begin{aligned} (j_s)_{\min} &= |j - j_L| && \text{if } h_1 - L \text{ is even,} \\ &= |j - j_L| + 1 && \text{if } h_1 - L \text{ is odd.} \end{aligned} \tag{4.15}$$

It is interesting to note that since the relations of the Bargmann–Moshinsky basis to the remaining known SU(3) ⊃ SO(3) basis have been discussed in the review of Moshinsky *et al* (1975), one only has to combine them with the results of the present paper to obtain the transformations from all known SU(3) ⊃ SO(3) bases to the canonical one.

Note added in proof. The right-hand side of (2.22) is proportional to $(\boldsymbol{w} \cdot \boldsymbol{w})^{(h_s/2 - j_s)/2} \mathcal{Y}_{j_s m_s}(\boldsymbol{w})$, where $\mathcal{Y}_{j_s m_s}$ denotes a solid harmonic, and $\boldsymbol{w}$ is a vector whose spherical components are defined by $w_{+1} = w_{11}/\sqrt{2}$, $w_0 = w_{12}$, and $w_{-1} = w_{22}/\sqrt{2}$.

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